ON THE LOGICAL STRENGTH OF THE AUTOMORPHISM GROUPS OF FREE NILPOTENT GROUPS

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ABSTRACT. Considering a particular case of a problem posed by S. Shelah, we prove that the automorphism group of an infinitely generated free nilpotent group of cardinality λ first-order interprets the full second-order theory of the set λ in empty language.

1. INTRODUCTION

In his paper [8] of 1976, S. Shelah suggested a general program of the study of the logical strength of first-order theories of the automorphism groups of free algebras. Recently he has again attracted attention to that program in his survey [9]. Namely, the problem 3.14 from [9] asks for which varieties $\mathbf{V}$ of algebras, letting F_λ for a free algebra in $\mathbf{V}$ with $\lambda \geqslant \aleph_0$ free generators, can we syntactically interpret in the first-order theory of $\mathrm{Aut}(F_\lambda)$ the full second-order theory of the set λ in empty language (possibly for sufficiently large cardinals λ). Recall that a theory T_0 in a logic $\mathcal{L}_0$ is said to be *syntactically interpretable* in a theory T_1 in a logic $\mathcal{L}_1$ if there is a mapping $\chi \to \chi^*$ from the set of all $\mathcal{L}_0$-sentences to the set of all $\mathcal{L}_1$-sentences such that

$$\chi \in T_0 \iff \chi^* \in T_1.$$

It should be pointed out that one the main results of [8] states that for *any* variety of algebras $\mathbf{V}$ the first-order theory of the *endomorphism semi-group* $\mathrm{End}(F_\lambda)$ syntactically interprets $\mathrm{Th}_2(\lambda)$ provided that a cardinal λ is greater than or equal to the power of the language of $\mathbf{V}$. The situation with the automorphism groups seems to be more difficult and the reason is obvious: despite being complicated in many cases, the endomorphism semi-groups of free algebras, as Shelah's analysis in [8] demonstrates, can be viewed as combinatorial objects.

There exist only a few examples of varieties for which the Shelah's problem is completely investigated: some varieties have the desired property (for instance, the variety of all vector spaces over an arbitrary division ring and the variety of all groups [10, 12]), some do not (the variety of all sets with no structure [6, 7], the automorphism groups of free algebras are here *symmetric*

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groups). To the best of the author's knowledge, there are no general results on the subject (however, Shelah introduces in [9, § 3] a wide class of so-called Aut-decomposable varieties, which are in many ways analogous to the variety of all sets).

The purpose of the present paper is to prove that the automorphism groups $\mathrm{Aut}(F_\lambda)$ of free groups F_λ in all varieties of nilpotent groups $\mathfrak{N}_s$ with $s \geqslant 2$ are logically strong enough to interpret by means of first-order logic the full second-order theory of λ for all infinite λ. We also consider a number of related questions; it is proved, in particular, that the first-order theory of the automorphism group of a finitely generated free nilpotent group of class $\geqslant 2$ is unstable and undecidable. The author is grateful to Oleg Belegradek, Edward Formanek and Alexandre Iwanow for helpful discussions.

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2. Reducing nilpotency class

Suppose that N is a free nilpotent group of class $s \geqslant 2$ and let $K_m(N)$, where m is a naturalnumber, denote the kernel of the homomorphism from the group $\mathrm{Aut}(N)$ to the group $\mathrm{Aut}(N/N_{m+1})$ induced by the natural homomorphism $N \to N/N_{m+1}$, from N to the free nilpotent group N/N_{m+1} of nilpotency class m. In particular, $K_1(N)$ is equal to $\mathrm{IA}(N)$, to the subgroup of so-called IA-*automorphisms* of N, and $K_s(N) = \{\mathrm{id}\}$.

Lemma 2.1. *Suppose that γ is an* IA-*automorphism. Then γ commutes with every element of the subgroup $K_m(N)$ modulo the subgroup $K_{m+1}(N)$.*

Proof. According to [1], the groups $K_m(N)$ form the lower central series of the group $K_1(N) = \mathrm{IA}(N)$; every element of an arbitrary group G commutes with the elements of the kth term of the lower central series of G modulo the $(k+1)$th term [4, Section 5.3]. □

Like in our previous papers [11, 13], any automorphism θ of N, which inverts all elements of some basis of N will be called a *symmetry.*

Lemma 2.2. *Let θ be a symmetry.*

(a) *Suppose that c is an element of N_m. Then θ either fixes c modulo N_{m+1} (when m is even), or inverts c modulo N_{m+1} (when m is odd);*

(b) *Suppose that γ is an element of $K_m(N)$. Then the conjugate of γ by θ either equals to γ modulo $K_{m+1}(N)$ (when m is even) or to the inverse of γ modulo $K_{m+1}(N)$ (when m is odd).*

Proof. (a) Assume that $\mathcal{B}$ is a basis of N such that θ sends each element of $\mathcal{B}$ to its inverse. Since the group N_m/N_{m+1} is abelian it suffices to prove that θ acts in a prescribed way on generators $[x_{i_1}, x_{i_2}, \ldots, x_{i_m}]N_{m+1}$, where

$x_{i_1}, \dots, x_{i_m}$ are elements of $\mathcal{B}$ [4, Section 5.3]. We have

$$\begin{aligned}\theta[x_{i_1}, [x_{i_2}, \dots, x_{i_m}]] &\equiv [{x_{i_1}}^{-1}, [x_{i_2}, \dots, x_{i_m}]^{(-1)^{m-1}}] \\ &\equiv [x_{i_1}, [x_{i_2}, \dots, x_{i_m}]]^{(-1)^m} (\mathrm{mod}\, N_{m+1}).\end{aligned}$$

(b) By (a). □

Suppose that φ is an involution from $\mathrm{Aut}(N)$ and $\varphi_1, \varphi_2, \dots, \varphi_m, \dots$ are arbitrary conjugates of φ. For every σ in $\mathrm{Aut}(N)$ let us construct the sequence

$$\{\sigma_m(\varphi_1, \varphi_2, \dots, \varphi_m) : m \in \mathbf{N}\}$$

of automorphisms of N as follows:

$$\begin{aligned}\sigma_0 &= \sigma, \\ \sigma_1 &= \varphi_1\sigma_0\varphi_1{\sigma_0}^{-1}, \\ \sigma_2 &= \varphi_2\sigma_1\varphi_2\sigma_1, \\ \sigma_3 &= \varphi_3\sigma_2\varphi_3{\sigma_2}^{-1}, \\ &\dots\end{aligned}$$

More formally, for every $m \geqslant 0$

$$\sigma_{m+1} = \begin{cases}\varphi_{m+1}\sigma_m\varphi_{m+1}{\sigma_m}^{-1}, & \text{if } m \text{ is even,} \\ \varphi_{m+1}\sigma_m\varphi_{m+1}\sigma_m, & \text{if } m \text{ is odd.}\end{cases} \tag{2.1}$$

The following result generalizes the corresponding fact from [13] proved there for free nilpotent groups of nilpotency class 2.

Proposition 2.3. *Let N be a free nilpotent group of nilpotency class s. Then an involution $\theta \in \mathrm{Aut}(N)$ is a symmetry modulo $\mathrm{IA}(N)$ (that is, coincides with some symmetry modulo the group $\mathrm{IA}(N)$) if and only if for every σ from $\mathrm{Aut}(N)$ and every tuple $\theta_1, \theta_2, \dots, \theta_s$ of conjugates of θ the automorphism $\sigma_s(\theta_1, \theta_2, \dots, \theta_s)$ of N is trivial.*

Proof. Suppose that $\theta = \theta^*\gamma$, where θ^* is a symmetry and γ is an IA-automorphism. Since θ is an involution, then $\theta^*\gamma = \gamma^{-1}\theta^*$. Any member θ_k of the tuple $\theta_1, \theta_2, \dots, \theta_s$, a symmetry modulo $\mathrm{IA}(N)$, also has the form $\theta^*\gamma_k$ for a suitable IA-automorphism γ_k.

Let us prove by induction on m that *the automorphism $\sigma_m = \sigma_m(\theta_1, \dots, \theta_m)$ is an element of $K_m(N)$*. This will follow the necessity part of the Proposition.

Indeed, if $m = 1$, then σ_m is an IA-automorphism, that is a member of $K_1(N)$. Assume that $\sigma_m \in K_m(N)$ and let m be, for instance, even. We have by Lemma 2.2(b) and Lemma 2.1:

$$\begin{aligned}\sigma_{m+1} = \theta_{m+1}\sigma_m\theta_{m+1}{\sigma_m}^{-1} &= {\gamma_{m+1}}^{-1}\theta^*\sigma_m\theta^*\gamma_{m+1}{\sigma_m}^{-1} \\ &\equiv {\gamma_{m+1}}^{-1}\sigma_m\gamma_{m+1}{\sigma_m}^{-1} \equiv \mathrm{id}(\mathrm{mod}\, K_{m+1}(N)).\end{aligned}$$

Let us prove the converse. It is well-known that every automorphism of the abelianization $\overline{N}$ of N, the free abelian group $N/[N,N]$, can be lifted up to an automorphism of N (see, for instance, [5, § 4] or [3, Section 3.1, Section 4.2]). Then it suffices to prove that for every involution of $\mathrm{Aut}(\overline{N})$, which is not $-\mathrm{id}$, there exist an infinite sequence of the form (2.1), constructed inside $\mathrm{Aut}(\overline{N})$, which contains no trivial members.

It can be seen quite easily that every involution $f \in \mathrm{Aut}(\overline{N})$, which is not $-\mathrm{id}$, has two f-invariant direct summands B, C of $\overline{N}$ with $\overline{N} = B \oplus C$ and $\operatorname{rank} B = 2$; moreover, the action of f on B can chosen so that $f|_B$ is neither id_B, nor $-\mathrm{id}_B$ ([13, Theorem 1.4], [2, Lemma 1]). This reduces the problem to the automorphism groups of two-generator free abelian groups; for the sake of simplicity we shall work with the group $\mathrm{GL}(2,\mathbf{Z})$.

According to the just mentioned result from [2], every involution in $\mathrm{GL}(2,\mathbf{Z})$ is conjugate either to the involution

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

or to the involution

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

hence the group $\mathrm{GL}(2,\mathbf{Z})$ has exactly two conjugacy classes of non-central involutions. One readily checks that for every integer m

$$\begin{pmatrix} 1 & 0 \\ 2m & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 2m-1 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.2}$$

where $\sim$ denotes the conjugacy relation.

Let S be a non-central matrix from $\mathrm{GL}(2,\mathbf{Z})$ and m an integer number. Suppose that

$$X(m) = \begin{pmatrix} 1 & 0 \\ 2m & -1 \end{pmatrix} S \begin{pmatrix} 1 & 0 \\ 2m & -1 \end{pmatrix} S,$$
$$Y(m) = \begin{pmatrix} 1 & 0 \\ 2m & -1 \end{pmatrix} S \begin{pmatrix} 1 & 0 \\ 2m & -1 \end{pmatrix} S^{-1}.$$

There are no difficulties in the verification of the following fact: some element of the 'general' matrix X (and Y) depends linearly on m. It follows that for a suitable integer m the matrix $X(m)$ ($Y(m)$) is again non-central. This means that, starting with a non-central matrix, we can construct an infinite sequence of the form (2.1) having no central matrices; in particular, there will be no trivial matrices in this sequence. Exactly the same argument, using matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 2m-1 & -1 \end{pmatrix},$$

proves the similar result for the second conjugacy class of non-central involutions in $\mathrm{GL}(2,\mathbf{Z})$. □

Corollary 2.4. *Symmetries modulo* $\mathrm{IA}(N)$ *form a definable family in the group* $\mathrm{Aut}(N)$.

Let $T^{-}(N)$ denote the set of all automorphisms $\{\sigma\}$ of N such that for every θ, which is a symmetry modulo $\mathrm{IA}(N)$, the conjugate of σ by θ is equal to σ^{-1}. Similarly $T^{+}(N)$ denotes the set of all automorphisms of N, which commute with every symmetry modulo $\mathrm{IA}(N)$.

Proposition 2.5. *Let N be a free nilpotent group of nilpotency class s. Then $K_{s-1}(N) = T^{+}(N) \cup T^{-}(N)$. Therefore $K_{s-1}(N)$, the kernel of a surjective homomorphism from* $\mathrm{Aut}(N)$ *to the automorphism group of a free nilpotent group of nilpotency class $s-1$ and of the same rank as one of N, is a definable subgroup of* $\mathrm{Aut}(N)$.

Proof. An arbitrary element σ from $T^{+}(N) \cup T^{-}(N)$ must commute with any product of two symmetries: if, for instance, $\sigma \in T^{-}(N)$, θ_1 and θ_2 are two symmetries then

$$\theta_1\theta_2\sigma(\theta_1\theta_2)^{-1} = \theta_1\theta_2\sigma\theta_2\theta_1 = \theta_1\sigma^{-1}\theta_1 = \sigma.$$

On the other hand, one finds among the automorphisms of N, which can be expressed as a product of two symmetries, conjugations (inner automorphisms of N) by primitive elements (that is, members of bases of N). This implies that σ commutes with every element of $\mathrm{Inn}(N)$. Hence σ preserves each element of N modulo the center of N. The center of N is equal to the subgroup N_s [3, Section 3.1], and therefore $\sigma \in K_{s-1}(N)$.

Let τ be a conjugation by a primitive element x of N. We are going to represent τ as a product of two symmetries. The element x is a member of some basis $\mathcal{B}$ of N. Suppose θ_1 is a symmetry, which inverts each element of $\mathcal{B}$. Then if a symmetry θ_2 is defined as follows

$$\begin{aligned}&\theta_2 x = x^{-1},\\ &\theta_2 y = x^{-1}y^{-1}x, \quad \forall y \in \mathcal{B} \setminus \{x\},\end{aligned}$$

the product of $\theta_1\theta_2$ of θ_1 and θ_2 is equal to τ.

Conversely, according to Lemma 2.2 (b) every element of $K_{s-1}(N)$ either lies in $T^{+}(N)$, or in $T^{-}(N)$. □

3. Interpretations

Theorem 3.1. *Let N be a free nilpotent group of class $\geqslant 2$. Then the automorphism group of N first-order interprets the automorphism group of a free nilpotent group of class 2 and of rank which is the same as one of N (uniformly in N).*

Proof. By Proposition 2.5. □

Until otherwise stated, we shall assume that N *is a free nilpotent group of class* 2 and that A *denotes the abelianization* $\overline{N}$ *of* N.

It can be shown that $\mathrm{Inn}(N)$, the subgroup of all conjugations, is a $\varnothing$-definable subgroup of $\mathrm{Aut}(N)$ [13, Corollary 3.2]. The group $\mathrm{Inn}(N)$ is

isomorphic to the free abelian group A. Thus, we can interpret in $\mathrm{Aut}(N)$ the free abelian group A and the automorphism group of A with the action on the elements of A.

We can also interpret in $\mathrm{Aut}(N)$ the family $\mathcal{D}$ of all direct summands of A with inclusion relation and a binary relation, say R such that

$$R(B, C) \longleftrightarrow A = B \oplus C.$$

One can prove that an involution f from $\mathrm{Aut}(A)$ is diagonalizable in some basis of A if only if there are no elements of order three in the set $K(f)K(f)$, where $K(f)$ denotes the conjugacy class of f (see proof of Proposition 2.4 in [13]). Hence the fixed-point subgroups of diagonalizable involutions can be used to interpret the direct summands. Having the group A interpreted in $\mathrm{Aut}(N)$, we can easily interpret the inclusion relation and the relation R on the family $\mathcal{D}$.

Summing up, we see that *the group* $\mathrm{Aut}(N)$ *first-order interprets the multi-sorted structure* $\mathcal{M}$ *with the following description:*

- the sorts of $\mathcal{M}$ are the free abelian group A, its automorphism group $\mathrm{Aut}(A)$ and the family $\mathcal{D}$ of all direct summands of A;
- all sorts carry their natural relations; the relations of $\mathcal{D}$ are the inclusion relation and the relation R;
- $\mathcal{M}$ has as one of the basic relations the membership relation on $A \cup \mathcal{D}$;
- there are relations defining the action of $\mathrm{Aut}(N)$ on other sorts.

Lemma 3.2. *Let A be of infinite rank. Then the first-order theory of the structure $\mathcal{M}$ syntactically interprets the full second-order theory of the set $|A|$ (in empty language), uniformly in A.*

Proof. It follows from the results in Section 4 of [8], that the first-order theory of the endomorphism semi-group $\mathrm{End}(A)$ of A syntactically interprets $\mathrm{Th}_2(|A|)$ (for the sake of convenience the reader may refer to [10], where the very similar case of varieties of vector spaces is considered in some details in the proof of Proposition 10.1; an analysis of the proof shows that it works also for free **Z**-modules, or, in other words, for free abelian groups).

To complete the proof, we could therefore interpret in $\mathcal{M}$ the endomorphism semi-group of A. There is a (folklore) trick by which the endomorphisms can be interpreted in structures similar to $\mathcal{M}$ constructed over modules. This trick can be briefly characterized as follows: three submodules, such that any two of them are direct complements of each other, are used to interpret the endomorphism semi-group of one of them. A detailed description of the trick for infinite-dimensional vector spaces can be found in [10] (see the proof of Proposition 9.3); the reader is again referred to [10] to see that everything works for free **Z**-modules as well. □

The following result solves the problem posed by S. Shelah (see the Introduction) for all varieties of nilpotent groups $\mathfrak{N}_s$, where $s \geqslant 2$.

Theorem 3.3. *The first-order theory of the automorphism group of any infinitely generated free nilpotent group N of class $\geqslant 2$ syntactically interprets the full second-order theory of the set $|N|$. The first-order theory of* $\mathrm{Aut}(N)$ *is therefore unstable and undecidable.*

Proof. By Theorem 3.1 and Lemma 3.2. □

We have also solved the problem of classification of elementary types of the automorphism groups of infinitely generated free groups from varieties $\mathfrak{N}_s$:

Theorem 3.4. *Let N_1 and N_2 be infinitely generated free nilpotent groups of the same class $\geqslant 2$. Then the automorphism groups* $\mathrm{Aut}(N_1)$ *and* $\mathrm{Aut}(N_2)$ *are elementarily equivalent if and only if the sets $|N_1|$ and $|N_2|$* (*with no structure*) *are equivalent in the full second-order logic.*

Proof. By Theorem 3.1 and Lemma 3.2. □

Let N again denote a free nilpotent group of class 2 (recall that A stands for the abelianization of N and $\mathcal{M}$ is the multi-sorted structure constructed over A).

We are going to estimate the logical strength/complexity of the first-order theory of $\mathrm{Aut}(N)$ in the case, when N is finitely generated.

Lemma 3.5. *Let A be of rank at least* 2. *Then the structure $\mathcal{M}$ first-order interprets* (*with parameters*) *the ring of integers* $\mathbf{Z}$.

Proof. Let us consider two direct summands B, C of A such that

$$A = B \oplus C \text{ and } \operatorname{rank} B = 2.$$

Write G for the group of all automorphisms of A which preserve B and point-wise fix C. Clearly, the structure $\langle G, B\rangle$ with natural relations (that is, with all relations on sorts along with the action of G on B) is isomorphic to the two-sorted structure $\langle \mathrm{GL}(2,\mathbf{Z}), \mathbf{Z}^2\rangle$ (taken in the same language as one of $\langle G, B\rangle$). It is a well-known and simple result that the latter two-sorted structure first-order interprets the ring of integers $\mathbf{Z}$. □

As an immediate corollary we have the following fact.

Theorem 3.6. *Suppose that N is a finitely generated free nilpotent group of class $\geqslant 2$. Then the first-order theory of the group* $\mathrm{Aut}(N)$ *is unstable and undecidable.*

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